Dynamic And Visuospatial Techniques Mediated By Technology For The Algebraic Activity

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Abstract
This paper describes the techniques for algebraic activity provided by AlNuSet. They have a dynamic and visuospatial nature that makes available tangible images of variable, algebraic expressions and propositions; such images encourage and support the exploration of properties peculiar to these algebraic objects.
The paper shows that the use of these techniques structure a new phenomenological space where algebraic objects, relations and phenomena are reified by means of representative events that fall under the visual, spatial and motor perception of students and teachers.
Moreover, the paper show that these techniques can be effectively used in the class to improve the teaching and learning of algebraic concepts
The theoretical framework of reference of this work is the instrumental approach used by many researchers to analyze the role of CAS in the learning of mathematics. In the paper the instrumented techniques of AlNuSet are compared with those of CAS, highlighting crucial differences for the teaching and learning of algebra.

1. Introduction
A crucial question of the teaching of algebra concerns the construction of appropriate conditions of mediation to share with students an idea of the objects that are the referent of the algebraic signs.
The algebraic objects have an abstract nature and students find it difficult to develop an idea of such objects through the signs that represent them [1], [5], [14], [15]. Sfard highlights that variables, expressions, propositions and algebraic functions are characterized by a dual nature: they can be understood both operationally and structurally. She also shows that these two conceptions are not mutually exclusive but complementary, and that the construction of a structural view of a mathematical object requires a preliminary development of operational competencies [16]. The development of operational competencies can occur through the solution of tasks by means of specific techniques centered on the use of the symbolic algebraic notation. The techniques used to solve the algebraic tasks, i.e. to solve an equation, to find the calculated values of an expression, to transform an expression etc., are a complex assembly of reasoning and routine work [2]. Each technique is characterized by a pragmatic value and by an epistemic value [6]. The pragmatic value of a technique is related to its effectiveness and efficiency in producing a socially acceptable result for the task at hand, as well as the limitations and the costs of its use [2]. The epistemic value of a technique is related to its capacity to activate questions and conceptual developments which make possible to understand what characterizes it structurally or justify it theoretically.

Research has shown that for many students, variables, expressions, propositions and algebraic functions, are only means for the solution of tasks and not objects characterized by specific properties [17]. The conceptualization of such entities as objects can occur through an epistemic interpretation of the techniques used to operate with them. However, the symbolic nature of these techniques makes it difficult to share with students their epistemic value. In the domain of algebra, differently from the geometric domain, visuospatial techniques for algebraic activity, able to provide tangible images of objects with which to operate and to encourage and support the exploration of the properties that characterize these objects, are not available. This can explain some of the difficulties that students have in the development of a structural conception of algebra.

This paper aims to present a new technology for teaching algebra that has been designed to support the development of this structural conception, the AlNuSet system [7] (A trial version of the software can be downloaded at the following address: http://www.alnuset.com. At this site also the guide of the software and demos area available). AlNuSet is an artifact that, thanks to the use of digital technology, makes available dynamic visuospatial techniques for algebraic activity. In this paper the key ideas that characterize the techniques instrumented in AlNuSet will be presented. Moreover, it will be highlighted that these techniques can foster a synthetic intuition of algebraic objects and the possibility to grasp their properties by observation. Furthermore, it will be evidenced that these dynamic visuospatial techniques can constitute an important support for understanding the epistemic value of the techniques used in current practice and that have a symbolic nature. The description of the techniques available in AlNuSet will be realized through a comparison with those of the Computer Algebra Systems (CAS) highlighting crucial differences for the teaching and learning of algebra.

2. **The rationale**

During the last 15 years a scientific debate on the role of technology to support teaching and learning processes in the domain of algebra has been going on. This debate originates from research studies carried out in different countries with the purpose of studying the use of Computer Algebra Systems (CAS) in school contexts. In particular, besides benefits [10], [19], obstacles and difficulties have been identified in using this technology by students and teachers [8], [9]. Results of these research works [3] highlight that the integration of CAS systems into the school practice of algebra remains marginal. To frame the results carried out by these research studies and the complexity of the processes involved in the educational use of CAS, some French researchers [2], [11] have elaborated a theoretical framework, named 'instrumental approach', integrating both the ergonomic theory [13] and the anthropological theory [6]. The ‘instrumental approach’ provides a frame for analyzing the processes of instrumental genesis both in their personal and institutional dimensions, and the effect of instrumentation issues on the integration of CAS in the educational practice. Using this framework, Artigue observes that CAS are extremely effective from a pragmatic standpoint and for this reason professionals (mathematicians, engineers…) are willing to spend time to master them [2]. At pragmatic level the effectiveness often comes with the difficulty to justify, at a theoretical level, the instrumented techniques used. In particular, this is true for users such as students that approach algebra, do not master mathematical knowledge and techniques involved in the solution of algebraic tasks. As a consequence, the epistemic value of the instrumented technique can remain hidden. This can constitute a problem for the educational context where technology should help not only to yield results but also to support and promote mathematical learning and understanding. In the educational practice, the techniques should manifest an epistemic value contributing to the understanding of objects involved. “*Making technology legitimate and mathematically useful from an educational point of view, whatever be the technology at stake, requires modes of integration that provide a reasonable balance between the* *pragmatic and the epistemic values of instrumented techniques*" ([4], p. 73). Even if the use of CAS seems fully legitimate in the class, the instrumented techniques cannot be institutionalized at epistemic level in the same way as paper and pencil ones. These results might account for the marginalization of CAS integration into the school practice for teaching and learning algebra. For some researchers, in order to integrate digital technology effectively in the domain of algebra, it is necessary to go beyond the experience of CAS and of their instrumented techniques and to face the design of new artifacts. As underlined by Monaghan, up to now CAS-in-education workers have paid little attention to design issues, preferring, in general, to work with the design supplied by CAS designers [12].

This article aims at pointing out techniques that can effectively support teaching and learning processes in algebra referring to AlNuSet (Algebra on the Numerical Sets), a system realized within the ReMath EC project to improve teaching and learning of crucial topics involved in the mathematical curricula such as algebra, functions and properties of numerical sets.
In the following sections, we first present some of the main algebraic techniques instrumented in AlNuSet and then we show, through some results of a teaching experiment, how these can be effectively used in the class to improve the teaching and learning of algebraic concepts.

3. The dynamic and visuospatial techniques of AlNuSet

Going beyond the design of CAS requires new creative ideas to embed in a new digital artifact, techniques for mathematical activity different from those of CAS. The advent of dynamic geometry software and of spreadsheets has evidenced that even a single creative idea makes it possible to characterize the mathematical techniques with new operational and representative dimensions (consider, for example, the techniques based on the drag of the variable point of a geometrical construction, as in the case of the dynamic geometry software, or the techniques based on the automatic re-computation of formulas of the table, as in the case of the spreadsheet). The instrumentation of a new mathematical technique changes its pragmatic and epistemic value and this can affect mathematics teaching and learning.

AlNuSet [7] is based on some ideas that characterize deeply the three environments composing it: the Algebraic Line environment, the Algebraic Manipulator environment, and the Function environment. These three environments make available instrumented techniques for the algebraic activity that have a visual spatial and dynamic nature and that are different from those of CAS.

3.1 Algebraic Line environment

The main idea that characterize the Algebraic line environment of AlNuSet is the representation of algebraic variables on the number line through sliding points associated to letters, namely points that can be dragged on the line with the mouse. The unit on the line between the letter (a label) and the sliding point associated to it is a dynamic visuospatial way to support the development of an idea of what a letter denotes in algebra. We note that the algebraic line of AlNuSet can be instantiated in different numeric sets choosing from Natural, Integer, Rational and Full Domain (Full domain: rational numbers extended to the rational powers of rational numbers).

The drag of the sliding point allows the students to experience what the variable denotes within the numerical set where it is used. In fact, with the drag of the sliding point associated with the letter, the point takes hold of their eyes and forces them towards the numeric elements which, within the chosen numeric set, the letter represents in an undetermined way and its interpretant, the sliding point, indicates in deictic and determined way. The unity between the letter and sliding point is the basic feature that transforms the number line into an algebraic line, and provides support for the development of an idea for the notion of variable quantity that is consistent with the definition of Euler, that is as a quantity "which is not determined or is universal, which can take on any value" (Euler, Introduction to analysis of the infinite). In the Algebraic Line environment the user can edit expressions with which intends to operate. The computer automatically computes the value of the expression on the basis of the value assumed by the variable on the line and associates the edited expression with a point on the algebraic line. When the user drags the sliding point of a variable, the computer refreshes the positions of the points corresponding to the expressions containing such a variable in an automatic and dynamic manner. The drag of the sliding point determines the dynamic movement of the expression on the line (see Fig. 1).

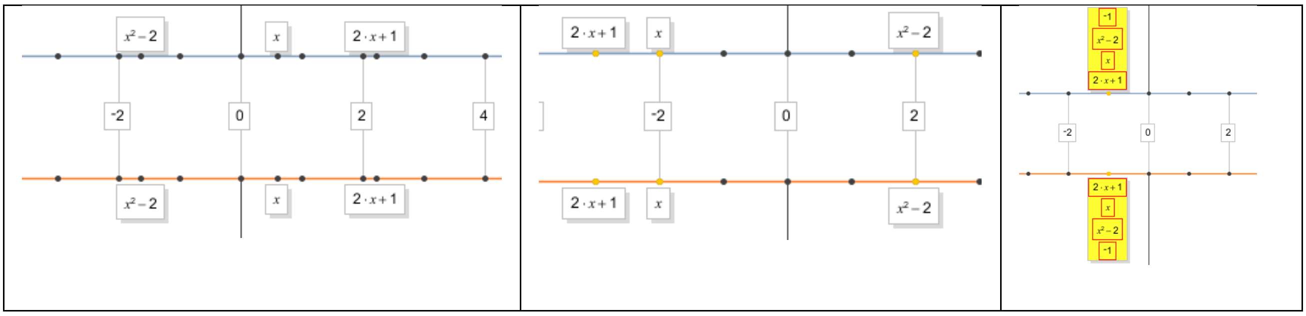


Fig. 1: Three images of the algebraic line dragging the sliding point x

Through the dynamic visuospatial technique of the dragging, the expression assumes the feature of an object that represents a class of numerical elements in an undetermined way. In fact, dragging the variable, the label of the expression moves on the line and represents all the values which the point associated to it, each time, indicates in a deictic way. The experience of dragging variable quantity on the line makes it possible to connect the notion of expression with that of function. In fact, the movement of the point associated with an expression on the line is dependent on the dragging of the variable. Through the dragging of the variable on the line, students can experience that the “function of a variable quantity is an analytic expression composed in any way from this variable quantity and numbers or constant quantities” (Euler) and can understand that notion of function is useful to express a quantitative change. The comprehension of the link between expression and function is crucial for the development of an idea of the indeterminacy of algebra [15].

The algebraic expressions are also objects that are characterized by specific properties and by relationships between them. Students need to build an idea of these objects and of their properties and relationships. For example, students must build an idea of what it means that two expressions are equivalent, opposite or reciprocal and must learn to recognize these relationships and justify them even at symbolic level. The table 1 shows some properties of expressions and of relationships between them that the technique of dragging makes possible to express in a dynamic visuospatial way.

The algebraic concepts of the first column of the table 1 find in the "gestaltic" perceptibility of the dynamics and visuospatial representations of the second column the conditions for their

Table 1: Some properties of expressions and of relationships between them expressed in a dynamic visuospatial way on the Algebraic Line of AlNuSet

Equivalent expressions	Two expressions are equivalent when they are associated to the same point on the line while dragging the variable on which they depend
Equivalence with restriction	Two expressions coincide with each other for each value assumed by the variable during its dragging on the line, except one or more values for which one of the two expressions "disappears" from the line (for example, the expression $(x + x^2) / x$ and the expression $x +1$ coincide for each value of x except $x = 0$ for which the first expression disappears from the line.)
Conditioned equality	The expressions are associated to the same point on the line only for certain values of the variable during its dragging on the line (for example the expressions 2x+3 and 5x are associated to the same point on the line only for x=1)
Opposite expressions	Two expressions are opposite when their respective points on the line are always symmetrical with respect to point 0 while dragging the variable on which they depend
Reciprocal expressions	Two expressions are reciprocal if the values of the variable for which an expression assumes value 1 on the line even the other expression assumes the same value 1.
$E_1(x)>E_2(x)\ \nabla\ x$	The point associated with $E_1(x)$ is always positioned on the line to the right of that associated with $E_2(x)$ while dragging the variable x
$E_1(x)>0\ \nabla\ x$	The point associated with $E_1(x)$ is always positioned on the line to the right of 0 while dragging the variable x
$E(x)=$ constant ∇ x	The point associated with the expression on the line does not move with the dragging of the variable x

understanding through observation. Moreover, the algebraic line environment has been designed to provide two very important instrumented techniques for the algebraic activity that present a dynamic and visuospatial nature: for identifying and validating the truth set of algebraic propositions and for finding the roots of polynomial with integer coefficients. To illustrate the educational importance of these two techniques, let us consider the inequality $x^2-2x-4\geq 0$, that once edited, is visualised in a specific window of this environment ("Set" window) and is associated to a marker (a little circle) that can be green or red. The solution of this inequality performed in the Algebraic Line environment is reported in Fig. 2. The process of solution is described below.

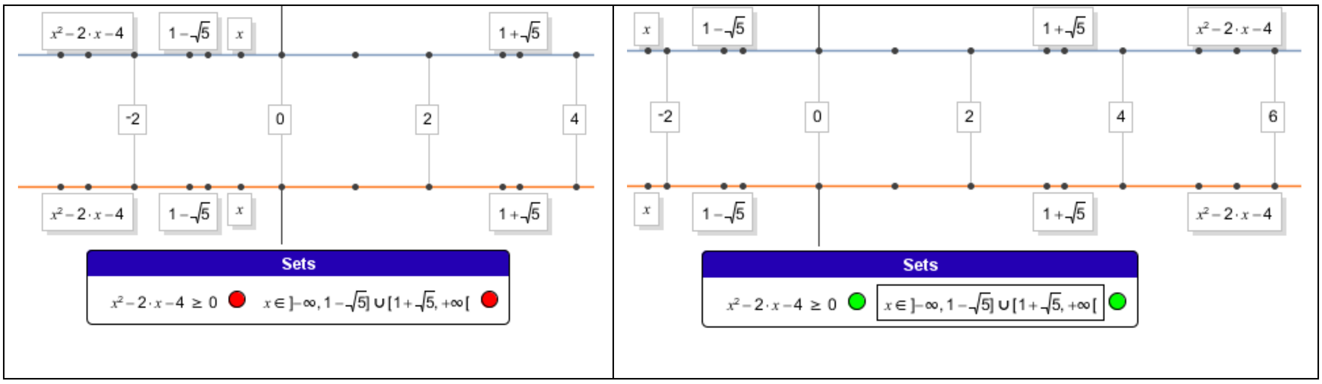


Fig 2: Solution of an inequality in the Algebraic Line environment

It is important to note that the marker associated with the inequality (and, in general, with any algebraic proposition edited in this environment) is its interpretant, which is under the control of the system and indicates the truth value of the inequality (green/true; red/false). Dragging the sliding point associated with the variable one can discover that there are values of x for which the inequality is satisfied. This discovery may occur through the observation of the position of the polynomial on the line while dragging the algebraic variable. The experience of the dragging allows the control of the values of x for which the expression is actually greater than 0, namely the polynomial is located on the right of 0 on the line. The discovery of values of x for which the inequality is satisfied may also be based on observation of the color of the marker while dragging the variable. The dragging allows the control of the values of x for which the inequality is true, namely the color of the marker associated with the inequality is green. However, since the extremes of the intervals that constitute the solution of the inequality are irrational numbers, to define exactly these intervals it is necessary to find the roots of the polynomial associated with the inequality. In the Algebraic Line of AlNuSet the root of a polynomial can be found by means of a specific instrumented technique. Even this technique is based on the dragging of the variable on the algebraic line. When through the drag of the variable a polynomial approximates the value of 0, an algorithm of the program determines the exact root of the polynomial and represents it as a point on the line. This technique dispenses the student from the need to make calculations; it can be controlled by the students through their visual and spatial experience. It is effective not only at a pragmatic level but also at an epistemic level, because it can concretely support the development of a discourse on the notion of root of a polynomial, as value of the variable that makes the polynomial equal to 0.

Once the roots of the polynomial associated to the inequality have been represented on the line, a graphic editor can be used to construct its truth set (see the figure). The technique involved in the definition of the truth set is visuospatial in nature: the graphical editor makes it possible to define the ranges of values of x where the inequality is true, by means of actions of selections with the mouse. The system automatically translates the defined range of values into the set notation. The formal definition of the set is associated with a marker (a little circle) whose color indicates whether the current value of the variable on the line belongs to this set or not.
The following table 2 shows how the dynamic and visuospatial characteristics of the described techniques make possible to interpret and validate properties of propositions and the relations between them.

Table 2: Properties of propositions and of relationships between them expressed in a dynamic visuospatial way on the Algebraic Line of AlNuSet

Interpretation and validation of the truth value of an algebraic proposition	The color of the marker associated with the algebraic proposition allows us to interpret and validate the truth value of the proposition in relation to the current value assumed by the variable on the line.
Interpretation and validation of the comparison conditions of the algebraic proposition	The position on the line of the expression(s) contained in the proposition allows us to interpret and validate the comparison indicated by the proposition in relation to the current value assumed by the variable on the line
Interpretation and validation of the truth set of a proposition	The concordance of color between the marker of the proposition and the marker of the set during the drag of the variable on the line makes possible to interpret and validate the defined numeric set as truth set of the proposition
Interpretation and validation of equivalent propositions.	The concordance of color between the markers of two propositions and the markers of their truth set during the drag of the variable on the line permit to interpret and validate the equivalence between the proposition

The visuospatial and dynamic techniques described in this section characterize the algebraic activity with AlNuSet and make them deeply different from that of the CAS, where these techniques are not available. They allow students to solve equations and inequalities through observation, without the need to perform algebraic manipulations. The techniques rely on tangible images of the abstract objects with which one operates. These images encourage the development of ideas about the abstract entities that are involved in the activity with algebraic propositions, and the possibility of finding an agreement on what is meant for the truth value and truth set of a proposition, for the algebraic solution for a (in)equation, for the role of unknown assumed by the letter in the solution of an equation, for the notion of (in)equations equivalent. Fostering a synthetic intuition of the algebraic objects and possibility to grasp their properties through observation, such techniques mediate the development of a structural conception of algebraic propositions and a more mature and abstract vision of the solution of equations and inequalities [17]. The experience with these techniques can be an important reference to assign an epistemic value to the techniques of symbolic nature of the current practice.

3.2 Functions environment

The main idea characterizing the design of the Functions component is the possibility to connect a dynamic functional relationship between variable and expression on the algebraic line with the graphical representation of the function in the Cartesian plane. The interface of this component has been equipped with the algebraic line and a Cartesian plane. Through a specific command, the expressions represented on the algebraic line can be automatically represented through a graphic in the Cartesian plane, as in the first image of Fig 3 concerning the expressions $E_1(x)= 2x+4$ and $E_2(x)= x^2-4$. Dragging the point corresponding to the variable on the algebraic line, two representative events occur: (i) on the algebraic line, the expression containing the variable moves accordingly; (ii) on the Cartesian plane, the point defined by the pair of values of the variable and of the expression moves on the graphic (see Fig. 3).

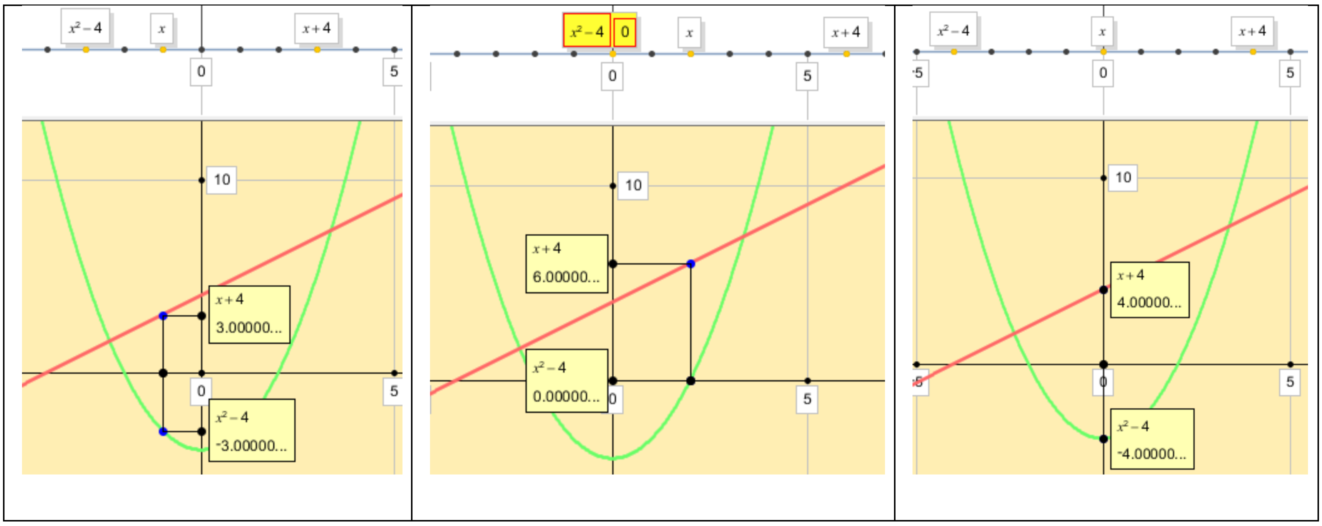


Fig. 3: Three images of the representative events emerging in the Function environment

The instrumented technique of the dragging of the sliding point on the line supports the integrated development of a static idea of function on the Cartesian plane with the dynamic idea of such a notion on the algebraic line. The integration of these two conceptions can be very useful to orient the interpretation of the graphics on the Cartesian plane and to develop an object-oriented conception of the algebraic functions [18]. For example, by dragging the sliding point associated with x on the algebraic line we can observe that:

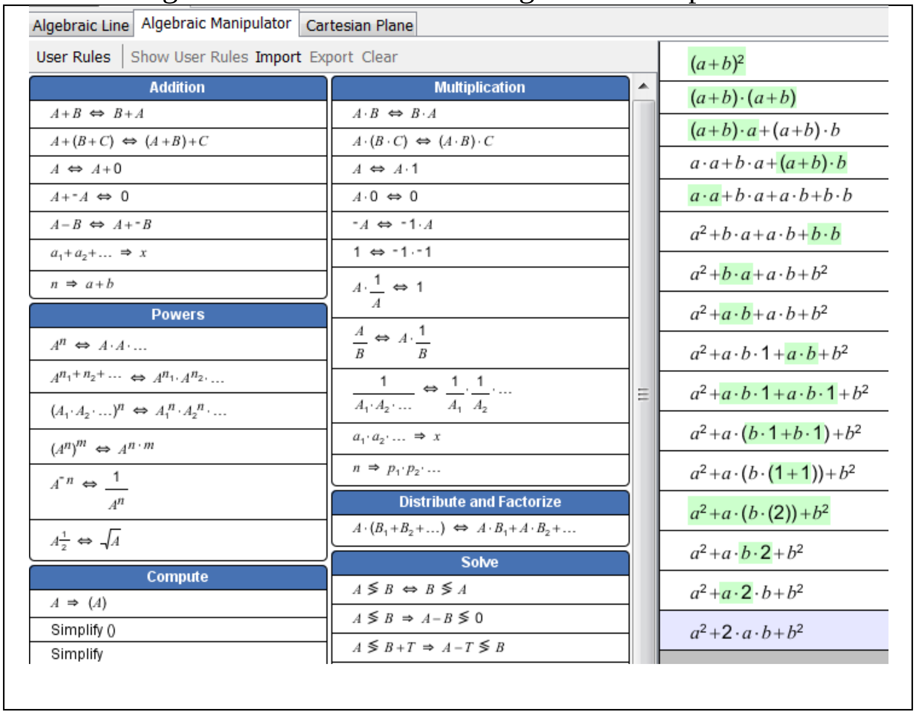
- the values of the variable x for which the graphs of $E_1(x)$ or $E_2(x)$ intersect the axis of abscissas, the expressions $E_1(x)$ and $E_2(x)$ on the algebraic line assume value 0;
- the values of the variable x for which the graph of $E_1(x)$ intersects the graph of $E_2(x)$, the expressions $E_1(x)$ and $E_2(x)$ on the algebraic line have equal value because are coincident in the same point;
- the values of the variable x for which the graphs of $E_1(x)$ or $E_2(x)$ are above the x-axis, the expressions $E_1(x)$ and $E_2(x)$ on the algebraic line have positive value, because they are located to the right of 0;
- the values of the variable x for which the graph of $E_1(x)$ is placed above the graph of $E_2(x)$, the value of $E_1(x)$ is greater than that of $E_2(x)$ because the expression of $E_1(x)$ on the algebraic line is located to the right of $E_2(x)$;
- …

All these observations are made possible by the integration of the dynamic representation of the expressions on the algebraic line with the representation of its corresponding function in the Cartesian plane. This integration makes the Function environment of AlNuSet deeply different from other environments for the representation of function in the Cartesian plane, including those based on the use of a slider.

3.3 Algebraic Manipulator environment

The main idea characterizing the Algebraic Manipulator of AlNuSet is the possibility to exploit pattern-matching procedures of computer science to transform algebraic expressions and propositions through a structured set of basic rules that are deeply different from those of the CAS. In CAS pattern matching procedures are exploited according to a pragmatic perspective oriented to produce a result of symbolic transformation that could be also very complex, as in the case of command like factor or solve. As a consequence, the techniques of transformation can be obscure for a non-expert user. In the Algebraic Manipulator of AlNuSet pattern-matching procedures are exploited to support the development of operational competencies and the construction of a structural conception of the algebraic transformation. The interface of this algebraic manipulator is reported in Fig. 4. The interface of this component has been divided into two distinct spaces: a space where symbolic manipulation rules are reported and a space where symbolic transformation is realized. The instrumented technique for algebraic transformation of this manipulator provides non-expert users with cognitive support in the development of the operational competencies of the algebraic transformation. A first support is the possibility to explore, through the mouse, the hierarchical structure that characterizes the expression or the proposition that one intends to manipulate. By dragging the mouse pointer over the elements of the expression or proposition at hand (operators, number, letters, brackets…), the system dynamically highlights, through the visual effect of a red box, the sub-expression defined by the element pointed. This feature allows the exploration of all the sub-expressions that characterize the expression at different levels of its hierarchical structure. This manipulator makes available a set of rules for the algebraic transformation that correspond to the basic properties of addition, multiplication and power operations, to the properties of equality and inequality, to basic logic operations among propositions and among sets. When a sub-expression is selected, only the transformation rules that can be applied on it are made active and highlighted in the interface. Once a rule of the interface has been selected, the computer automatically re-writes the expression or the proposition according to the rule of transformation applied.

Fig. 4: The interface of the Algebraic Manipulator



In this manipulator, the operational knowledge for the algebraic transformation is embedded in the functioning of the interface. The student, through interaction with it, can learn to transform algebraic expressions by observing and experiencing what rules can be applied on a selected sub-expression and the results that they produce.
Moreover a fundamental function of this component allows the student to create a new transformation rule (user rule) once this rule has been proved using the rules available on the interface that correspond to the axioms of basis of the algebraic transformation. For example, once the rule of the remarkable product has been proved, it can be added as new user rule in the interface and it can successively be used in other transformation, as described above.
This feature allows a deductive structural approach to the algebraic transformation centered on the use of some basic axioms (properties of operations) and the demonstration of progressively more complex rules for the algebraic transformation (theorems). This approach contributes to assign an epistemic value to the algebraic manipulation techniques used in current practice.

4. Teaching experiment

In this section, we show how the dynamic and visuospatial techniques of AlNuSet can be effectively used to improve the teaching and learning of Algebra in class. In particular, we focus on techniques that support students to grasp meaning of algebraic concepts. For this aim we use data collected from a teaching experiment carried out in a class of 22 students of the second year of upper secondary school (15-16 years old). The experiment lasted 6 weeks, with sessions of 1 hour and half per week. In table 3 topics treated in each session of the teaching experiment are presented. In the third column of the table the main dynamic visuospatial techniques used for each session are reported. The techniques used in the experiment make reference to those reported in table 1 and table 2. The last column in the table 3 highlights the scholastic level in which tasks were

Table 3: Sessions of the teaching experiment reporting the educational topic, the techniques of the AlNuSet's environment used in each session and the school level of reference for each session.

Session	**Educational Topic**	**AlNuSet's environment and techniques used**	**Scholastic level in which tasks were experimented**
1	What Algebraic expressions are?	Algebraic Line: dragging of the variable sliding point *to observe* the value of expressions and the relationships between expressions on the line (Tab 1).Techniques of the Algebraic Manipulator	Lower and Upper Secondary school
2	Construction of an idea for opposite expressions and equivalent expressions	Algebraic Line: dragging of the variable sliding point *to observe* the value of expressions and the relationships between expressions on the line (Tab 1)	Lower and Upper Secondary school
3	Construction of an idea for the algebraic notions of conditioned equality and identity	Algebraic Line: dragging of the variable sliding point to identify and validate proprieties and relationships reported in table 2. Techniques of the Algebraic Manipulator	Lower and Upper Secondary school
4	What is an equations with parameters	Algebraic Line: properties expressed in table 2 for equations Techniques of the Algebraic Manipulator	Upper Secondary school
5	Construction of ides for Inequations and inequations with parameter	Algebraic Line: properties expressed in table 2 for inequation Techniques of the Algebraic Manipulator	Upper Secondary school
6	Construction of an idea of algebraic functions and their properties	Function environment: dragging of the variable sliding point *to observe* the relationship between an expression on the line and its graphic on the Cartesian plane.	Upper Secondary school

successfully experimented. Actually, it is interesting to observe that some of the tasks presented in this teaching experiment, were also used in another teaching experiment of the lower secondary school. As a matter of fact, the most of techniques used in the experiment are accessible also to students attending lower scholastic levels and support even at that level the comprehension of mathematical concepts that are usually very difficult to grasp for students in a standard educational practice.

For each session a set of tasks was prepared. Students worked alone to solve each task; at the end of each task a discussion was usually required from the teacher. The aim of the discussion was double: to verify the solutions constructed by the students, and to institutionalise the new acquired concepts. In this report we present some results of the sessions 1, 3 and 6. We have chosen these 3 sessions because they allow us to show how each of the three environments of AlNuSet can be used in the class. Furthermore, the discussions developed in these three sessions allow us to show how dynamic and visual spatial techniques of the AlNuSet environments can really improve the teaching and learning processes of algebra.

<u>Session 1</u>

Students had not used AlNuSet previously. The teacher presented AlNuSet showing some specific technical features of the software. Then she distributed a paper containing some tasks that in this phase served also to allow students to familiarise with the system. In the first tasks students were request to put in formula some algebraic properties expressed in verbal form such as: the consecutive of the triple of *x*, the quadruple of *x* increased by 3, etc. Moreover, students were requested to verify their answers inserting the expressions on the Algebraic Line and using the technique of dragging of the variable sliding point.

As we had expected all students were able to put in formula the properties expressed in verbal form (*3x*, *4x+3*, etc.), but they were not aware of what the constructed expression could indicate, and of the dependence relationship that exists between expression and variable. Only when they could move the variable on the Algebraic line and could observe the effect that this movement produces on the constructed expression, they become aware of these important aspects characterizing the algebraic expressions ("*the expression 3x assumes only values that are multiples of 3... it probably takes all multiple of 3*").

An interesting aspect observed during this first exploration was that when students had to move the expression *3x*, most of them failed because they directly moved the expression and not the variable *x*. Here is an example:

> Francesca: 3x does not move
> Danilo: We are not able to move 3x
> Teacher: Why isn't it moving?
> Francesca: we are trying but with no results…
> Teacher: Why are you moving directly 3x?
> Francesca: Because I want to move the expression 3x
> Teacher: You should not move directly 3x because this expression is dependent on the variable x
> *Silence*
> Teacher: In which way can you move 3x?
> *Silence*
> Teacher: you have to move x

Danilo: x??
Danilo moves x
Danilo: Ahh... 3x is dependent on x
Francesca: Ahhh… it was really too difficult for us…
Danilo: But we can understand…3x is dependent on x, so it moves only if x moves…

The possibility to move the variable on the line to see the effect on the expression was an important dynamic and visual spatial technique to grasp that: an expression describes not only a process of calculation but also a quantitative change [17]; the letter contained in an expression indicates the elements of the numerical set in which the expression is defined [14]; there is a dependence relationship between the value of the variable and those of the expression [17]; the way in which the values of the variable and of the expressions change (covariance) is determined by the structure of the expression [18].
Some techniques of the Algebraic Line also allow the teacher to propose interesting tasks to students usually hard to solve (without AlNuSet). Let’s see as example the last task of the session 1.

The teacher asks a student to carry out the following computation: *Think a number, double it, add 6, divide the result by two, and subtract from it the number that you thought initially.* The teacher says: *The result is 3* The teacher proposes the exercise to two other students changing the number to add. At the end she proposes the following exercise: If **x** is the initial number, and **a** is the number to add, write an expression to translate the described computation. Represent the expression on the Algebraic Line. What can you observe when you move **x**? What can you observe when you move **a**? What is the result of the expression? Represent the result on the AL and check your hypothesis

The solution of this task on the Algebraic Line is very effective to understand the different meaning between variable and parameter. Observe that the expression $\frac{2*x+a}{2}-x$ is equivalent to the expression $\frac{a}{2}$.
As a consequence, the teacher may always guess the result of the expression because it is not dependent on the value thought by the student.
In the Algebraic Line, the expression $\frac{2*x+a}{2}-x$ does not move dragging the variable *x* (see Fig. 5). In this way, students can do the experience that this expression is not dependent on *x*. Students may also observe that the value of this expression is always the half of the value of *a* (which is the value that the teacher requested to add).
When the teacher asks to carry out the following computation, all students are really surprised when she guesses the results of the calculus “*Think a number, double it, add 6, divide the result by two, and subtract from it the number that you thought initially*”.
The teacher proposes the same task modifying the number to be added to two other students. She asks to add *4* and then *8*.
The students are not able to explain why the teacher is able to guess always the result. The teacher asks to write an expression to translate the computation.
She asks to insert the correct expression on the Algebraic Line and move alternatively *a* and *x* to observe the behaviour of the expression.

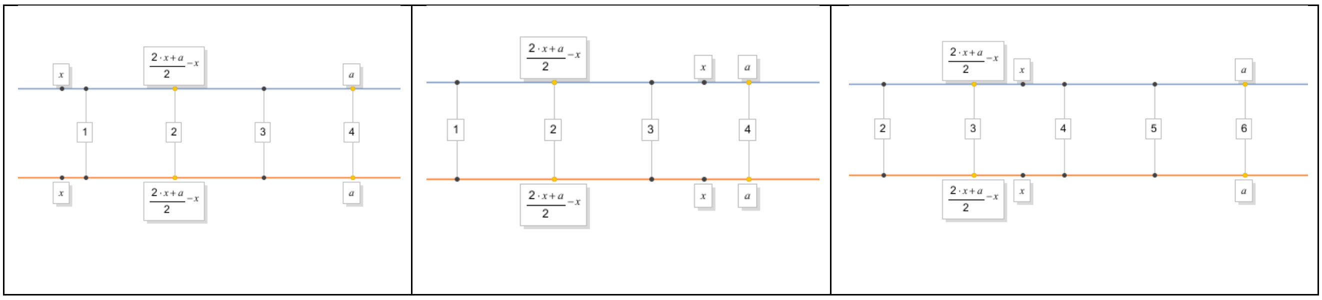
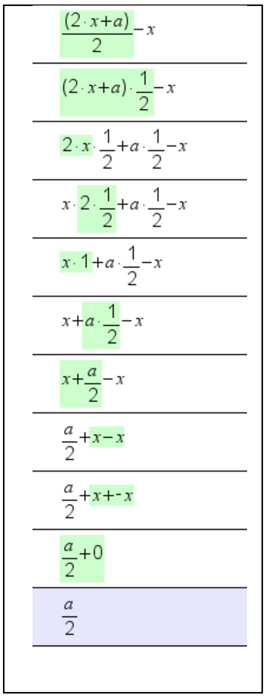


Fig. 5: The solution of the last task, session1, in the Algebraic Line. Moving *x* the expression does not move. Moving *a*, the expression move and its value is the half of the *a* value.

In the following a part of the discussion developed in the class at the end of the work of the task, is presented.

Teacher: What does it happen when you move ***x***?
Alberto: the expression does not move!
Fabrizio: nothing happens!
Teacher: are you surprised?
A lot of students: yesss
They continue to move x
Teacher: and when you move ***a***?
Fabrizio: there are some values that are ok...
Giuseppe: but not all values... sometimes the expression disappears
Students speak among them to try to explain why sometimes the expression disappears
Sara: the expression is dependent only on ***a***
Teacher: This is an important point! We have seen that if we move x this expression does not move accordingly. This expression does not depend on *x*. On the contrary, we have seen that moving ***a*** the expression has a particular behaviour. Who can account for this behaviour?
Carlo: wait... if ***a*** is an even number then I can see the expression, but if ***a*** is an odd number, the expression disappears! I cannot see it!!?
Dylan: The expression is exactly the half of ***a***
Teacher: are you sure that it is always the half of ***a***?
Federico: yes it is true, it is the half of ***a***
Sara: this is why you guessed the results of the expression... It is always the half of the number you required to add..
A lot of students: yess! It is true!!!
Teacher: but why, in your opinion, the expression disappears when *x* is an odd number?
Carlo: because an odd number divided by *2* is a decimal number
Alberto: it is a decimal number
Carlo: You asked to add only even numbers...otherwise it wouldn't have been possible to divide by two
Teacher: Perfect! In a previous case, in the computational task I asked to add *6*, then *4* and then *8*. They are even numbers. I can calculate the half of an even number. This is why I could guess the result. Is it clear to all?... But what happens if we change the domain?
Danilo: the expression does not disappear
Alberto: it is always the half of ***a***, we cannot see it in the Integer numbers because the integers are not decimal numbers!
Teacher: So the expression is always equal to…
All students: the half of a
Teacher: so we can write
The teacher writes on the blackboard $\frac{2*x+a}{2}-x=\frac{a}{2}$

Teacher: can you prove it? Try to prove it in the manipulator.



On the left the required proof.

Obviously, the intervention of the teacher was necessary to construct the proof. It was the first time that students use the Algebraic Manipulator. However, the integration between the Algebraic manipulator and the Algebraic Line was very useful for the teacher to explain students the equivalence between expressions. The teacher can explain why an expression is equivalent to the transformed expression in the manipulator, coming back to the algebraic line where all the expressions belong to the same point under each movement of *a*. All the expressions do not move under each movement of *x*. For students this was an important experience because they could justify what does it means that the first expression is equivalent to the last one. Students could now clearly understand why the teacher was able to guess the results of the calculus required to students at the beginning of the task.

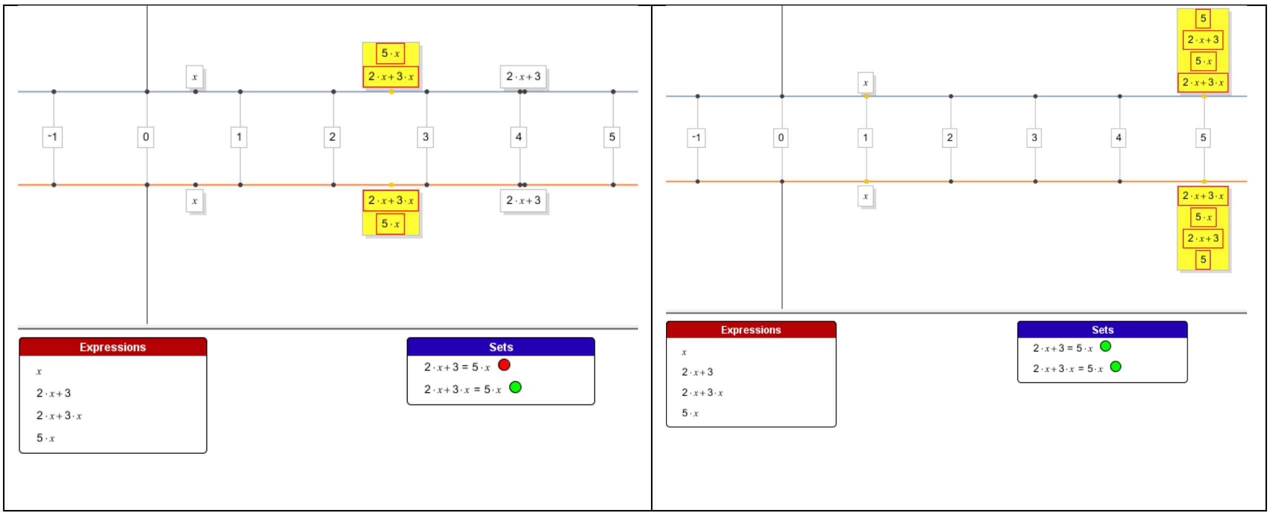
<u>Session 3</u>

The aim of this session was to reflect on meanings involved in the solution of an equation. Among the different tasks that have been proposed to students in the teaching experiment we have chosen that under reported because it was very interesting and effective for the our educational aim. When this task was proposed, students were already able to solve first grade equation from a technical point of view but they were not aware of the meanings involved in their solution.

> Consider these two equalities:
> *2*x+3=5*x*
> *2*x+3*x=5*x*
> Write everything you are able to say about them.
>
> To check your answer, insert x and the following expressions on the AL:
> *2*x+3; 2*x+3*x;* *5*x*
> Then edit the two equalities.
> What can you observe moving *x*? Justify

No student was able to provide an acceptable answer regarding the two equalities. Most students wrote that the first equality (2x+3=5x) is not correct, while the second one (2x+3x=5x) is correct. Two students wrote that the two equalities are equal. Four students did not write any answer. When they verified their answer on the Algebraic Line, the difference between the two equalities became evident (as shown in figure 6). In fact, moving *x*, they can observe that: the two expressions *2x+3x* and *5x* have always the same value because they coincide to the same point on the line (and hence these two expressions are equivalent); the two expressions *2x+3* and *5x* have the same value only when *x* is dragged on the point 1. Moreover, the color of the marker associated with the algebraic proposition *2x+3x=5x* is green for each movement of the variable x while the color of the marker associated with the proposition $2x+3=5x$ is green only when x is positioned on the point 1 (when x is moved on the other points on the line, the marker associated to this proposition is red). These dynamic visual spatial techniques allow students to connect the two equalities to the notion of equation and to experience two important notions involved in the solution of an equation, namely the notions of conditioned equality and of identity.

Fig. 6: The solution of the task, session 3, in the Algebraic Line



The students' answers to the task after the experience with AlNuSet are completely different in respect to their first answers. A typical answer was: "*2x+3x is always equal to 5x while 2x+3 is equal to 5x only when x is 1. So 2x+3x=5x is always true while 2x+3=5x is true only when x=1*". These results were very outstanding for the teacher. It is really unusual for students to be able to justify the difference between conditioned equality and identity.

Session 6

During this last session, the teacher proposed to her students a task to be solved in the Function environment. Previously students had used this environment to study the sign of a linear function and of a quadratic function. In this task students were required to insert on the algebraic line *x*, *a* and the expression ax^2. Then students were required to represent ax^2 in the Cartesian plane considering before *x* as variable and then *a* as variable. The task was the following:

Consider the expression ax^2 According to you is this expression always positive? Represent this expression on the Cartesian plane. If **x** is defined as a variable what do you obtain? If **a** is defined as a variable what do you obtain? Consider **x** as a variable and **a** as a parameter. What happens when you move the parameter **a** on the line? Consider **a** as a variable and **x** as a parameter. What happens when you move the parameter **x** on the line?

The aim of this task was to introduce students to the concepts of parameter and variable in working with functions. In the Function environment if $a*x^2$ is represented considering x as variable, the graphic of a parabola appears in the Cartesian plane. Moving x on the line, the corresponding points on the graphic are visualised accordingly. Moving a on the line the shape of the parable changes, because a runs as parameter in this function.
On the contrary, when the function ax^2 is represented considering a as variable a line is represented on the plane. Moving a, the correspondent points on the line are visualised, while moving x the line changes (the angular coefficient changes).
Through this task, students could experience that the same expression on the line can correspond to two graphs in the Cartesian plane, namely to two different functions. Exploiting the visuospatial and dynamic features of this environment, students could understand why this occurs and could construct an idea of the two different roles and meaning that a letter can assume in a function (variable and parameter).
Let's see, in the discussion developed at the end of this session, how students are able to justify the behaviour of the functions, through the mediation dynamic and visual spatial techniques of the system.

Teacher: so what can you say about the expression ax^2? Is it always positive?
Students: noooo... It depends
Giuseppe: If the variable is x, I obtain a parabole
Federico : that is concave if **a** is positive
Carlotta: and convex if **a** is negative
Federico: and it is 0 if **a** is 0
Teacher: in this case **x** is
Students: variable
Giuseppe: and **a** is parameter.
Teacher: and if we consider **a** as a variable?
Irene: it is completely another thing... If a is variable we find a line
Carlotta: that is positive when **a** is positive and negative if **a** is negative
Fabrizio: and x is the parameter
Alberto: x is the slope of the line... Moving x the slope changes
Alberto: in other words...if I represent $\boldsymbol{ax^2}$ in respect to the variable x, I obtain a parable and moving x I find all the points on the parabole, while moving **a,** I change the shape of the parabola... In opposition if I represent ax^2 in respect to the variable **a**, I find a line and moving **a** I find all the points of the line, while moving x, I change the slope of the line.

The reported discussion gives an idea of the way in which students become aware of some difficult concepts through the mediation of the dynamic and visual spatial techniques of AlNuSet.
The distinction between variable and parameter is usually one of the most difficult algebraic concepts to be grasped from students. Notwithstanding this, students were able to construct fine justifications about these concepts even in their copies.

5. Conclusion

In this paper the main ideas that characterize the environments of AlNuSet have been described.
It has been shown that the visuospatial and dynamic techniques available in these environments make possible to operate with algebraic variable, expressions, propositions and algebraic function in a different way with respect to the CAS.
The techniques of AlNuSet are particularly useful from an educational point of view. They can be easily integrated into school practice because they provide a reasonable balance between the

pragmatic level and the epistemic one. From the pragmatic point of view AlNuSet techniques make it possible to deal effectively with any kind of task of the algebra curriculum and in many cases provide support for the development of operational skills involved in solving certain types of tasks (for example, in reading and interpretation of graphs or in the development of skills in algebraic transformation).

However, it is at the epistemic level that the instrumented techniques of AlNuSet show a great educational importance. The teaching experiment has highlighted that the activity with the AlNuSet techniques facilitates the understanding of what a variable, a parameter, an expression, a proposition or an algebraic function are , and encourages the exploration of the properties that characterize these objects. At the end of the experiment students were able to justify some learned concepts without the support of the system. For example, they were able to explicit algebraic properties used during the transformation of an expression even if it was not explicitly required from the teacher. This was a very important result. In the current educational practice of algebra it is really difficult for students to become aware of the properties (commutative, property, distributive property, etc.) that are involved in an algebraic transformation and that underlie the rules of symbolic manipulation used by them .

Weak students with difficulties in treating and acquiring mathematical concepts, were able to solve tasks in AlNuSet and they showed important progress in their learning of algebra.

Furthermore, one of the main results of the experiment was that teacher could refer to algebraic concepts and their properties (parameters, variable, expression, function, etc.) not in abstract or in implicit way. AlNuSet was for the teacher an educational tool that she could use to allow her students to make a concrete experience of these important algebraic concepts and to support her communication with students on these concepts. These results were observed in each teaching experiment made with the use of AlNuSet and hence they assume a general validity.

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